Fourier analysis of equivariant quantum cohomology III

2024年1月25日 22:08

(Reduction Conjecture) X : T-variety , $Y = X /\!\!/ T$: GIT quotient

$$J_X^{eq} \in \mathcal{L}_X^{eq} \overset{\text{discrete FT}}{\rightsquigarrow} I = \sum_{[\beta]} \kappa\left(\hat{S}^{-\beta} J_X^{eq}\right) \hat{S}^{\beta} \in \mathcal{L}_Y$$

?

- This holds for the projective spaces (toric varieties)

<u>Ex</u> $V \longrightarrow B$ vector bdle s.t. $V^{\vee}$ is generated by global sections ⇝ well-defined non-equivariant quantum cohomology

$S^1 \curvearrowright V$ fiberwise scalar action ⇝ $V /\!\!/ S^1 = \mathbb{P}(V)$

<u>Thm</u> (I-Koto) J_V^{λ} : equivariant J-function

The discrete Fourier transform

$$I = \sum_{k=0}^{\infty} \frac{q^k}{\prod_{c=1}^{k} \prod_{\substack{\delta : \text{Chern roots} \\ \text{of } V}} (\delta + p + cz)} J_V^{p+kz} \quad \text{lies in } \mathcal{L}_{\mathbb{P}(V)}$$

q : Novikov variable associated with a line in $\mathbb{P}(V)$

$p := c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}(V))$

Rem This is a generalization of Elezi-Brown mirror thm (when V is a direct sum of line bdles)

Sketch of proof $\exists\ V \hookrightarrow \underline{\mathbb{C}^N}$ embedding (by assumption)

$\mathbb{P}(V) \hookrightarrow B \times \mathbb{P}^{N-1}$ cut out by a section of $\underbrace{\left(\mathbb{C}^N/V\right) \otimes \mathcal{O}(1)}_{\text{convex bdle}}$

Use quantum Riemann Roch of Coates - Givental

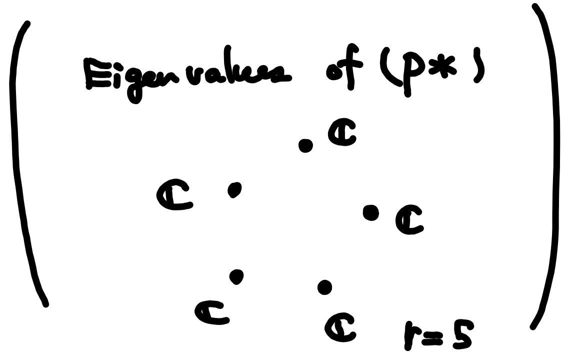
Ex X : $\mathbb{C}^\times$-variety $Y \subset X$: $\mathbb{C}^\times$- invariant divisor with normal wt ± 1

$\rightsquigarrow$ reduction conjecture holds for $Y = X /\!/ \mathbb{C}^\times$

§ Decomposition of QDM

$QH^*_{\text{small}}(\mathbb{P}^{r-1}) \cong \mathbb{C}[p,q]/(p^r - q)$ $\underline{\text{semisimple}}$ for $q \neq 0$

direct sum of $\mathbb{C}$ as a ring



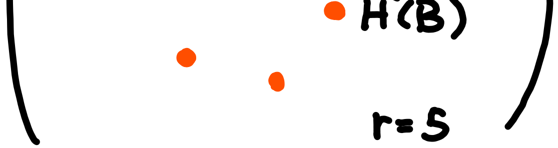
$QH^*_{\text{v.s.}}(\mathbb{P}(V)) \cong H^*(B)[p,q]/(p^r + c_1(V)p^{r-1} + \cdots + c_r(V) - q)$

vertical small

(count only curves

$\cong \bigoplus H^*(B)$ for $q \neq 0$

contracted by
$\mathbb{P}(V) \to B$



After full quantum deformation

$$QH^*(\mathbb{P}(V)) \cong \bigoplus QH^*(B) \ ?$$

Thm (Koto-I) $\quad V \to B$ as above

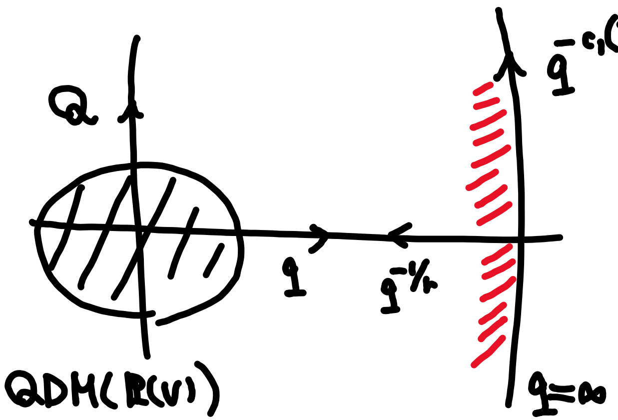
$\exists$ formal invertible map $\sigma = (\sigma_j)_{j=0}^{r-1} : H^*(\mathbb{P}(V)) \to \bigoplus_{j=0}^{r-1} H^*(B)$

$\exists$ isom $\Phi : QDM(\mathbb{P}(V)) \cong \bigoplus_{j=0}^{r-1} \sigma_j^* QDM(B)$ preserving the conn & pairing

- (Q, q) : Novikov var of $\mathbb{P}(V)$
  $\uparrow$ base $\quad \uparrow$ fiber
  $(Q, q)^d = Q^{\pi_* d} q^{p \cdot d}$

- σ is defined over $\mathbb{C}((q^{-1/r}))[\![Q]\!]$
  Φ is defined over $\mathbb{C}[z]((q^{-1/r'}))[\![Q]\!]$
  $r' = \begin{cases} r & r : \text{odd} \\ 2r & r : \text{even} \end{cases}$

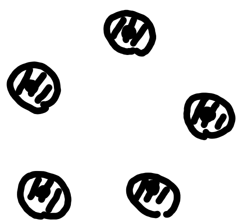
- decomposition happens near $q = \infty$



$q^{-c_1(V)/r} Q = Q_B$

- asymptotics of σ_j

$$\sigma_j \Big|_{Q=\tau=0} \sim r\, q^{1/r} \zeta_r^j + O(q^{-1/r})$$



$QH^*(B)$

$r=3$ case

- Since $z\nabla_v \to (v*)$ (Eigenvalues of $E*$)

$$z\nabla_v = z\partial_v + (v*) \quad \text{as } z\to 0,$$

σ gives an isom of quantum cohomology F-manifolds (with Euler vector field)

i.e. $\quad d\sigma : \left(H^*(\mathbb{P}(V)), *_\tau\right) \xrightarrow{\cong} \bigoplus_{j=0}^{r-1} \left(H^*(B), *_{\sigma_j(\tau)}\right)$

§ Blowup case

X : smooth proj var, $Z \subset X$: smooth subvariety of codim r

$\tilde{X} = \mathrm{Bl}_Z X$

$$\boxed{\begin{array}{l}\textbf{Thm} \quad \exists \text{ formal invertible map } \left(\tau, (\sigma_j)_{j=0}^{r-2}\right) : H^*(\tilde{X}) \longrightarrow H^*(X) \oplus \bigoplus_{j=0}^{r-2} H^*(Z) \\ \qquad\quad \exists \text{ isom } \Phi : \mathrm{QDM}(\tilde{X}) \cong \tau^*\mathrm{QDM}(X) \oplus \bigoplus_{j=0}^{r-2} \sigma_j^*\mathrm{QDM}(Z)\end{array}}$$

Rem • We need an extension of the Novikov ring to a common ring

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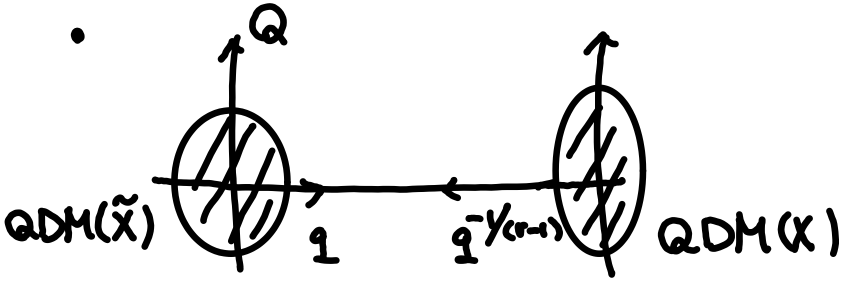
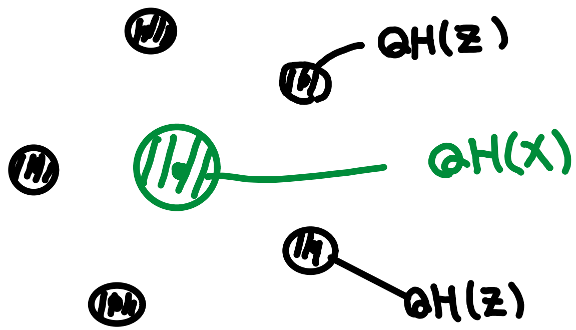


$$\begin{cases} \tau(\tilde{\tau})\big|_{Q=\tau=0} = 0 \\ \sigma_j(\tilde{\tau})\big|_{Q=\tau=0} = (r-1)(-q)^{-\frac{1}{r-1}} + O(1) \end{cases}$$

q: line in the exceptional locus as $q \to \infty$

Q: other Novikov variables

- Eigenvalues of $(E \star_{\tilde{\tau}})$ near $Q = \tilde{\tau} = 0$, $q \to \infty$



explain this case

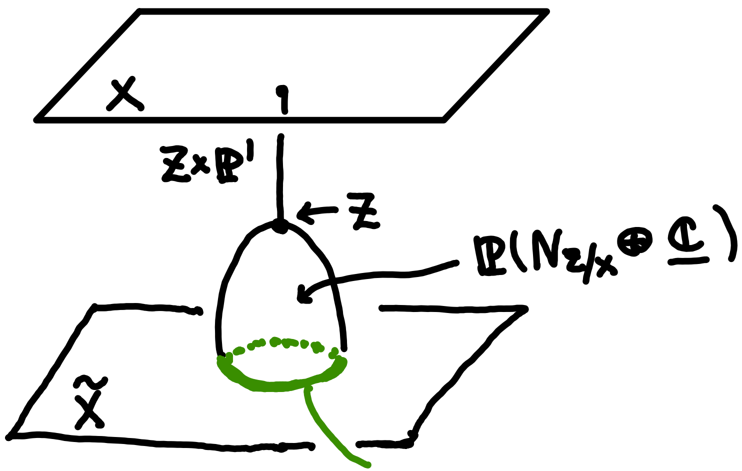
§ Strategy of the Proof

: similar for $\mathbb{P}(V)$ and blowups

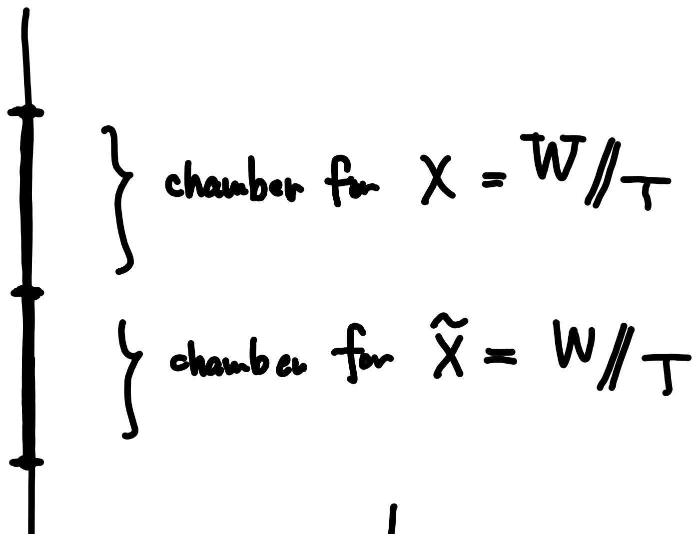
Write this as a GIT variation

$$W = \mathrm{Bl}_{Z \times \{0\}}(X \times \mathbb{P}^1) \curvearrowleft T_{\mathbb{C}} = \mathbb{C}^\times \text{ action on the 2nd factor}$$

"master space"

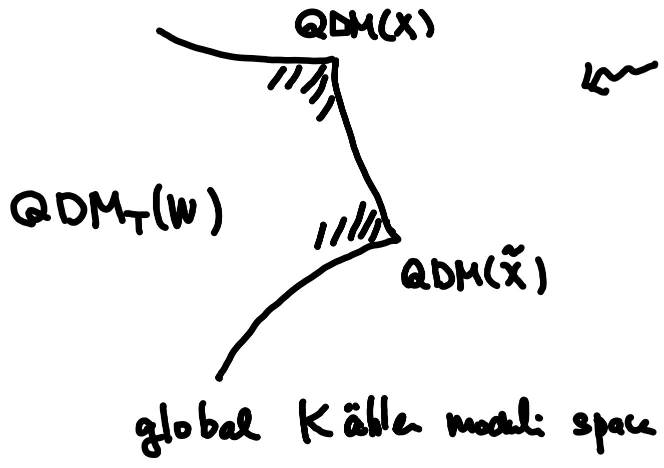


$\xrightarrow{\quad}$ μ moment map for T-action

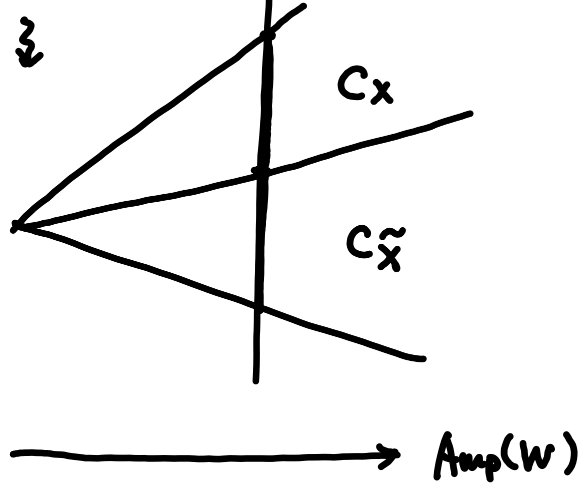


$\mathbb{P}(N_{Z/X})$

$$W^T = X \sqcup Z \sqcup \tilde{X}$$



There are 2-chambers in $C_T(W)$ (T-ample cone)



Rem $\dim H_2^T(W) = \dim H_2(X) + 3$

$= \dim H_2(\tilde{X}) + 2$

we have 2 redundant Novikov variables

(1) We construct maps of D-modules

$$\begin{tikzcd} & QDM_T(W) \arrow[r, "F_X"] \arrow[dl, "F_Z"'] \arrow[dr, "F_{\tilde{X}}"] & QDM(X) \\ QDM(Z)^{\oplus(r-1)} & & QDM(\tilde{X}) \end{tikzcd}$$

$F_X, F_{\tilde{X}}$: discrete / continuous FT

(we prove the reduction conjecture in this case)

F_Z : continuous FT

2-kinds of FT we use :

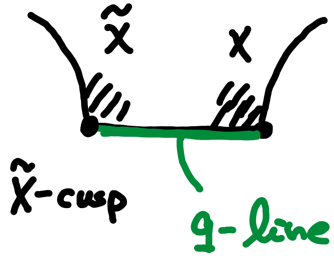
discrete FT ... GIT quotients (reduction conj) continuous FT ... fixed locus (this works by Coates – Givental thm)

- For X or $\tilde{X}$, we can show "discrete FT = conti FT" and the reduction conj follows

(2) Consider the submodule $\mathbb{C}[C^\vee_{\tilde{X},M}]\cdot QDM_T(W) \subset QDM_T(W)[Q^{-1}]$

(not $\otimes_{\mathbb{C}[NE^T_M(W)]} \mathbb{C}[C^\vee_{\tilde{X},M}]$: the pull-back)

and take its graded completion $QDM_T(W)^\wedge_{\tilde{X}}$ at the $\tilde{X}$-cusp



(3) Show the isomorphism

$$QDM(X)\oplus QDM(Z)^{\oplus(r-1)} \xleftarrow[F_X\oplus F_Z]{\cong} QDM_T(W)^\wedge_{\tilde{X}} \xrightarrow[F_{\tilde{X}}]{\cong} QDM(\tilde{X})$$

Check along the q-line

Compare the fiber at the $\tilde{X}$-cusp

- The decomposition preserves the pairing

(Rem $\deg q = 2(r-1) > 0$ plays an important role (blowup is discrepant))

§ Continuous FT associated with a fixed locus

- twisted GW invariants (Coates–Givental)

  $E \to X$ vector bundle with fiberwise scalar T-action
  (T acts trivially on X)

- universal stable map

$$\begin{tikzcd} \mathcal{C}_{0,n,d} \arrow[r, "f"] \arrow[d, "\pi"] & X \\ X_{0,n,d} & \end{tikzcd}$$

$X_{0,n,d}$ = moduli sp of $g=0$ stable maps

- (e_T^{-1}, E)-twisted GW invariants are defined by replacing $[X_{0,n,d}]_{\text{vir}}$ with

$$[X_{0,n,d}] \cap e_T^{-1}(E_{0,n,d})$$

$$E_{0,n,d} := R\pi_*(f^* E)$$

$\rightsquigarrow \begin{cases} (e_T^{-1}, E)\text{-twisted quantum product} \\ = \quad \text{fundamental solution} \\ = \quad \text{Givental cone } \mathcal{L}^{tw} \end{cases}$

Thm (Coates–Givental)

$$\mathcal{L}^{tw} = \Delta \cdot \mathcal{L},$$ where Δ is the operator given by the asymptotic exp

$$\Delta^{-1} \sim \prod_{\delta} \frac{1}{\sqrt{-2\pi z}} (-z)^{-\delta/z}\, \Gamma\left(-\frac{\delta}{z}\right) =: G_E$$

δ: T-equiv Chern roots of E

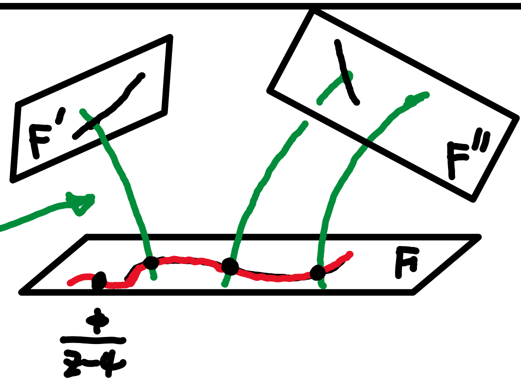
Thm (Givental, Brown, Fan-Lee)

X : smooth proj variety with T-action. $F \subset X^T$: T-fixed component

$$f \in \mathcal{L}_X^{equiv} \implies f|_F \in \mathcal{L}_F^{tw}$$ twisted by $(e_T^{-1}, N_{F/X})$

In particular $J_X^{eq}|_F$ lies in $\mathcal{L}_F^{tw}$ (We need the Laurent expansion at $z=0$)

☺ virtual localization



1-dim orbits

$$\left\langle \frac{\phi}{z-\psi}, \text{branch}, \text{branch}, \text{branch} \right\rangle_F^{tw}$$

Hence "$G_{N_F} \cdot \underbrace{J_X^{eq}|_F}_{\text{in } \mathcal{L}_F^{tw}} \in \mathcal{L}_F$" $\rightsquigarrow$ we will actually consider

$$\int e^{\lambda \log S/z}\, G_{N_F} \cup J_X^{eq}|_W \, d\lambda$$

• Suppose $T = S^1$ take a splitting of $0 \to H_2(X) \to H_2^T(X) \to H_2^T(pt) \to 0$ given by the fixed component F

σ_F

$H_2^T(pt) = \mathbb{Z} \ni 1 \longleftrightarrow \mathcal{S} = \hat{\mathcal{S}}^{\sigma_F(1)}$

shift op

Lem

$$H^*_{\hat{T}}(X)_{loc} \ni f \longmapsto G_{N_F} \cdot f|_F$$

$$\boxed{\text{intertwines } \mathcal{S} \text{ with } e^{-z\partial_\lambda}}$$

difference eqn $\Gamma(1+z) = z\Gamma(z)$ of the Γ-function

$\rightsquigarrow$ Hence the Fourier transformation

$$\boxed{\underset{H^*_{\hat{T}}(X)_{loc}}{f} \mapsto \int e^{\lambda \log S/z} \underset{H^*(F)}{G_{N_F} \cdot f|_F} \, d\lambda}$$

Mellin–Barnes integral

should intertwine $\begin{cases} \mathcal{S} \text{ with } S \\ \lambda \text{ with } z S \frac{\partial}{\partial S} \end{cases}$

- We can define the formal asymptotics of the integral using the Stirling approximation

$$G_{N_F} \sim \frac{1}{\sqrt{e_T(N_F)}} e^{-\frac{1}{z} \overbrace{\sum_\alpha \operatorname{rk}(N_{F,\alpha})(\alpha \log \alpha - \alpha)}^{W(\lambda)}} (1 + O(z))$$

(Note : $\alpha \in \mathbb{Z}\lambda$)

$$\rightsquigarrow \int e^{\lambda \log S/z} G_{N_F} \cdot f|_F \, d\lambda \sim \int e^{(\lambda \log S - W(\lambda))/z} \frac{1}{\sqrt{e_T(N_F)}} (1 + O(z)) f|_F \, d\lambda$$

phase function

twisted superpotential

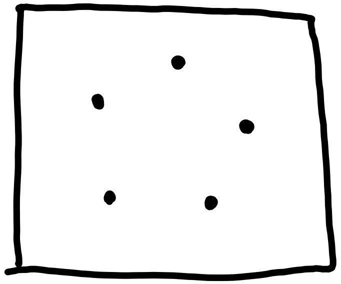
$$\varphi(\lambda) = \lambda \log S - W(\lambda)$$

- Expand the integrand in Taylor series at the critical pt λ_0 of $\varphi(\lambda)$ and perform termwise Gaussian integral

$n = \operatorname{codim} F$

- If T-weights of N_F are $\{w_1\lambda, \cdots, w_n\lambda\}$

$$\Rightarrow \varphi(\lambda) = \lambda \log S - \sum (w_i\lambda \log(w_i\lambda) - w_i\lambda)$$



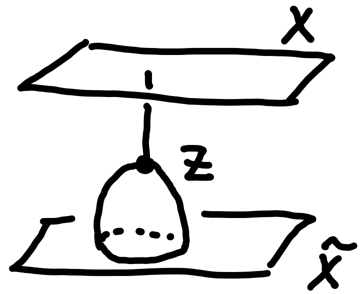
$$\Rightarrow \quad \lambda_0 = \left(\frac{s}{\prod_i w_i^{w_i}} \right)^{1/w} \quad \text{if } w := \sum_i w_i \neq 0$$

w many asymptotics

- formal asymptotic expansion belongs to $\mathcal{L}_F$ by Coates–Givental thm

$\rightsquigarrow$ defines a map $\mathcal{L}_X^{\mathrm{equiv}} \xrightarrow[FT]{} \mathcal{L}_F$, $\quad f \mapsto \int e^{\lambda \log s / z} G_{N_F} f|_F \, d\lambda$

a map $QDM_T(X) \longrightarrow QDM(F)$



__Ex__ $W = Bl_{Z \times \{0\}}(X \times \mathbb{P}^1)$ has Z as a fixed component

with normal T-weights $(1, \underbrace{-1, \cdots, -1}_{r})$ $\rightsquigarrow$ $(r-1)$ many critical pts

__Rem__ For the other fixed components $X, \tilde{X}$, continuous FT = discrete FT

(reduction conj)

☹ residue computation

$\square$